

## Singular Solutions to a Partial Differential Equation for Computer Imaging

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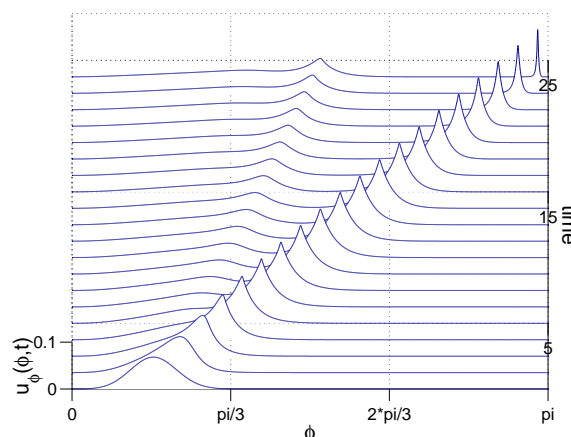
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Certain problems in computer imaging involve the dynamics of smoothly deforming one image to another. The dynamics can be looked at as a flow on the group of smooth, invertible maps, i.e., the diffeomorphism group. The variable of interest is the velocity  $\mathbf{u}(\mathbf{x}, t)$ , which describes the rate of deformation of the image. These flows have been studied extensively when taken with respect to the kinetic energy norm  $\int |\mathbf{u}|^2$ , as this yields the Euler equations for an ideal fluid. Another norm, the  $H^1$  norm  $\int |\mathbf{u}|^2 + |\nabla \mathbf{u}|^2$ , is relevant for problems in computational anatomy [1]. We have studied the dynamics with respect to this norm on manifolds with symmetry, including the sphere and hyperbolic space. This is a first step in understanding the dynamics of the diffeomorphism group on non-Euclidean spaces with respect to norms other than the kinetic energy norm; for example, Sharon and Mumford [2] recently proposed the space known as Teichmüller space as a framework for the study of 2D-shape analysis.

The partial differential equation (PDE) we study is called *EPDiff*, for Euler-Poincaré equation on the diffeomorphism group. It describes geodesic motion with respect to the choice of Hamiltonian. As stated above, this is Euler's equation for an ideal fluid when the Hamiltonian is the kinetic energy; and for motion on the real line, with the  $H^1$  norm as the Hamiltonian, we recover the Camassa–Holm equation for shallow water waves.

EPDiff has remarkable singular solutions where the velocity has the form of a solitary wave. A solitary wave is a waveform where the velocity is some kind of bulge over a small part of the domain and rapidly decaying to zero away from that bulge. For example, the Camassa–Holm equation



*A smooth initial velocity breaks up into puckons.*

has solitary wave solutions of the form

$$u(x, t) = p(t) e^{-|x - q(t)|}.$$

We see that  $q(t)$  marks the position of the wave's peak and  $p(t)$  gives the peak height, or the wave's velocity. Remarkably, the parameters  $q, p$  satisfy Hamilton's canonical equations. When one considers a space that is more complicated than the real line, similar types of solitary wave solutions exist with canonical parameters  $q, p$ , but the waveforms are more complicated than  $\exp(-|x|)$  and less amenable to analytical study.

We have studied the dynamics of EPDiff (with the  $H^1$  norm) on manifolds which possess a symmetry with respect to  $S^1$  rotations. The most familiar example is the sphere, and hyperbolic space is also a manifold of this type. Recall that on the real line, the solitary wave had its peak at a point  $x = q(t)$ . Analogously, on the sphere, the solitary wave solution will have a peak at a certain latitude  $\phi = \Phi(t)$  so that the peak extends all the way around the sphere at that latitude. Thus the wave is like a girdle on the sphere. In reference to a quote by Puck in *A Midsummer Night's Dream*, "I'll put a girdle about the earth in forty minutes," we call these solitary waves *puckons*. When an initially smooth velocity evolves under EPDiff, it interestingly breaks up into a series of the solitary waves. The figure above shows this numerically. Therefore, understanding the behav-

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ior of these solitary wave solutions is essential to understanding the dynamics of EPDiff.

A puckon also has another degree of freedom: rotation. The girdle can move up and down the sphere, and it can also rotate with a conserved angular momentum. For a single puckon, we have found the analytical solution for its canonical colatitude  $\Phi(t)$ , which is sinusoidal with a period depending on the value of the Hamiltonian. In addition, the rotating puckon is constrained to move between minimum and maximum latitudes which depend on the puckon's angular momentum and the Hamiltonian. This is illustrated in the numerical solutions shown in the column to the right. (Only the solutions for the meridional velocity  $\mathbf{u} \cdot \hat{\phi}$  and the canonical colatitude  $\Phi(t)$  are shown here.) Also, a point on the girdle of the rotating puckon follows a great circle of the sphere. But if the angular momentum is zero, then the puckon has no minimum and maximum latitudes, and it will collapse upon itself at either pole and bounce back into the midlatitudes.

In addition to the sphere, we also studied the specific case of hyperbolic space in some detail. See [3] for more information.

## Co-investigator

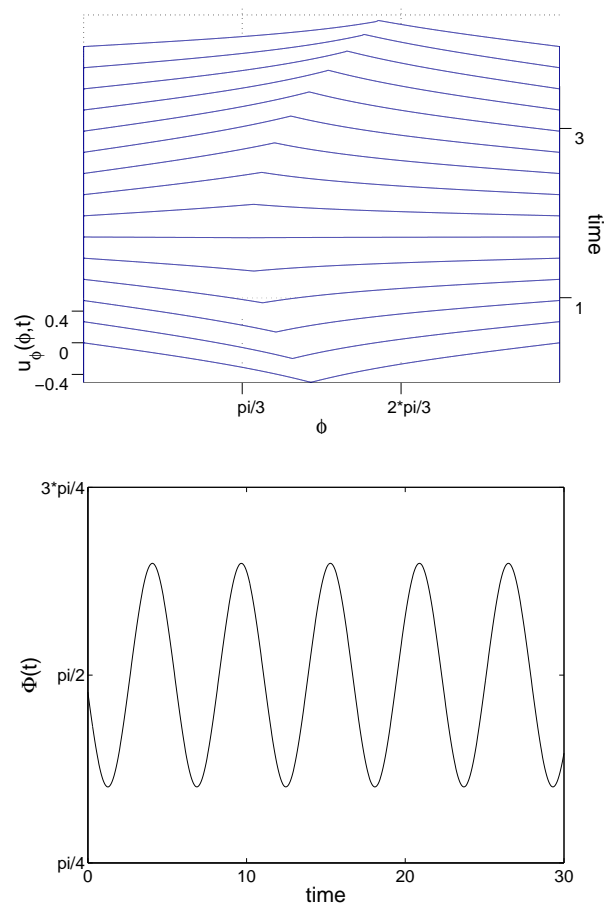
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## References

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*A single rotating puckon. Numerical solution to the EPDiff PDE (top) and numerical solution to the canonical equations for  $\Phi(t)$  (bottom).*